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ON THE LIMIT CARRYING CAPACITY OF A PIPE-LINE CROSS-SECTION

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Abstract—The object of the paper is to analyse the limit state of a tube subjected to quadruply-combined non-symmetric loadings: internal pressure, torque, bending moment and axial force. The system of all sixteen equations of the theory of plasticity is reduced to three by introducing two stress functions ϕ, ψ and plastic modulus H. It is solved by means of triple perturbation method under the assumption of small influence of bending, twisting and tension on the load carrying capacity of the tube subjected to internal pressure.

1. INTRODUCTION

PROBLEMS of thick-walled tubes subjected to combined loadings in the plastic range have been given much attention in technical publications. However the problem of the simultaneous existence of pressure, bending moment and torque which is so important for practical applications has not as yet been solved; all three-dimensional pipe lines are as a rule exposed to just this kind of combined loading. The problem belongs to the category of non-symmetrical problems of the theory of plasticity. Solution of a general circularly symmetrical case: axial force, internal or external pressure, torque as well as tangential pressures—axial and circumferential, was given by Życzkowski [9]. A detailed analysis, followed by deriving effective formulae for stress distribution and limit surface for cases of three-fold combined load, i.e., torsion, tension and pressure difference has been carried out by Skrzypek and Życzkowski [6]. The limit carrying capacity of a thick-walled tube in a similar case of load was also studied by Panarelli and Hodge [3]. This however, was limited to general formulae in integral form and numerical calculations for a few well-known individual cases.

Mrowiec and Życzkowski [1] have discussed the elastic carrying capacity of a thick-walled pipe-line subjected to internal pressure and bending moment. A complete solution for the plastic range considering the effect of bending as well as tension upon the limit capacity of a tube subjected to internal pressure has been given by Życzkowski [11].

Problems of simultaneous bending or torsion and tension of bars of various cross sections have been discussed by Życzkowski [10] and Wnuk [8].

Using the method of small parameter, Piechnik [4] has given effective solutions covering the effect of bending upon the limit state of a twisted bar of circular cross section, and Piechnik and Życzkowski [5] have solved the reverse case of the effect of torsion on the capacity of bars subjected to large bending. Classification and a detailed review of solutions of combined loadings problems in the theory of plasticity is given by Życzkowski [12].

This paper—generalizing the paper [11]—aims to analyze the case of four-fold combined load of thick-walled tube: internal pressure, bending moment, axial force and torque in plastic state, as well as to obtain solutions by means of the extended method of small parameter.

2. ASSUMPTIONS

In this paper we shall deal exclusively with the analysis of a purely plastic state, namely, assuming full plastification of the entire cross section of the tube. This assumption being fulfilled in particular cases (pure bending, tension, internal pressure or torsion) is not strictly retained in the case of combined load. In fact, when reaching the limit carrying capacity, some parts of the section still remain in the elastic state. In consequence, it leads to a discrepancy in fulfilling the boundary condition at the inner radius of the tube. These discrepancies however are small and will be discussed in detail later on.

Hencky–Ilyushin's or Levy–Mises' theory of plasticity will be applied in our solution (with purely formal substitution of strain velocities instead of strains).

The material is assumed to be perfectly plastic, incompressible and obeying the Huber–Mises–Hencky's yield condition.

Our considerations will be carried out in cylindrical coordinates r, θ, z (Fig. 1), remembering that in our case the stresses and strains are independent of the z coordinate.

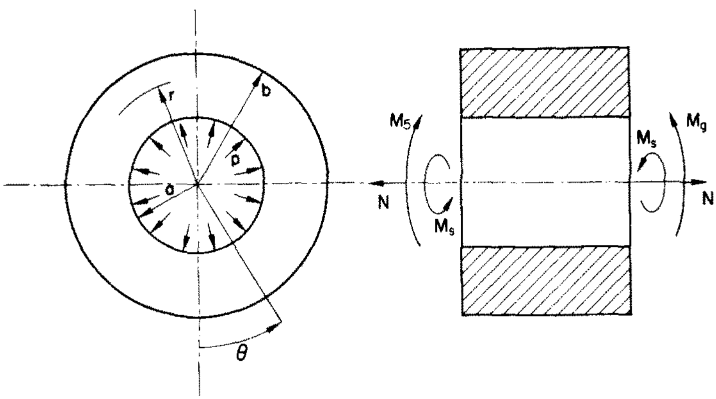

FIG. 1

With the above assumptions all 16 equations of the theory of plasticity will be used: equations of equilibrium

$$\frac{\partial\sigma_r}{\partial r}+\frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta}+\frac{\sigma_r-\sigma_\theta}{r}=0,$$

$$\frac{\partial\tau_{rz}}{\partial r}+\frac{1}{r}\frac{\partial\tau_{\theta z}}{\partial\theta}+\frac{\tau_{rz}}{r}=0, \tag{2.1}$$

$$\frac{\partial\tau_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta}+\frac{2\tau_{r\theta}}{r}=0;$$

yield condition

$$(\sigma_r-\sigma_\theta)^2+(\sigma_\theta-\sigma_z)^2+(\sigma_z-\sigma_r)^2+6(\tau_{r\theta}^2+\tau_{\theta z}^2+\tau_{rz}^2)=2\sigma_0^2; \tag{2.2}$$

σ_0 being the yield point at uniaxial tension, physical relations between stresses and strains

$$\begin{aligned} \varphi(\sigma_r-\sigma_\theta) &= \varepsilon_r-\varepsilon_\theta, \\ \varphi(\sigma_\theta-\sigma_z) &= \varepsilon_\theta-\varepsilon_z, \\ 2\varphi\tau_{\theta z} &= \gamma_{\theta z}, \\ 2\varphi\tau_{r\theta} &= \gamma_{r\theta}, \\ 2\varphi\tau_{rz} &= \gamma_{rz}; \end{aligned} \tag{2.3}$$

φ is the variable plastic modulus; ε_{ij} stands here for strains (Hencky–Ilyushin) or strain velocities (Levy–Mises), respectively;
incompressibility condition

$$\varepsilon_r+\varepsilon_\theta+\varepsilon_z = 0; \tag{2.4}$$

compatibility conditions [2]

$$\begin{aligned} \frac{\partial^2\varepsilon_z}{\partial r^2} &= 0, \\ \frac{\partial^2}{\partial\theta\partial r}\left(\frac{\varepsilon_z}{r}\right) &= 0, \\ \frac{1}{r^2}\frac{\partial^2\varepsilon_z}{\partial\theta^2}+\frac{1}{r}\frac{\partial\varepsilon_z}{\partial r} &= 0, \\ \frac{1}{r^2}\frac{\partial^2\varepsilon_r}{\partial\theta^2}-\frac{1}{r^2}\frac{\partial^2}{\partial r\partial\theta}(r\gamma_{r\theta})+\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varepsilon_\theta}{\partial r}\right)-\frac{1}{r}\frac{\partial\varepsilon_r}{\partial r} &= 0, \\ \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r\gamma_{\theta z})\right]-\frac{\partial^2}{\partial r\partial\theta}\left(\frac{1}{r}\gamma_{rz}\right) &= 0, \\ \frac{\partial^2}{\partial\theta^2}\gamma_{\theta z}-\frac{\partial^2}{\partial r\partial\theta}(r\gamma_{\theta z}) &= 0. \end{aligned} \tag{2.5}$$

3. BASIC EQUATIONS

The solution of the set of equations (2.1)–(2.5) is possible owing to the fact that the first three compatibility conditions determine uniquely the strain ε_z

$$\varepsilon_z = C_1+C_2 r\cos(\theta-\theta_0). \tag{3.1}$$

Introducing the dimensionless radius $\rho = r/b$, parameters proportional to curvature κ, and axial extension λ, respectively, and remembering that for the system of coordinates on Fig. 1 the angle $\theta_0 = 0$, the final result is

$$\varepsilon_z = \lambda+\kappa\rho\cos\theta \tag{3.2}$$

Let us now assume, dimensionless stress functions ϕ and ψ so that the conditions of internal equilibrium (2.1) are fulfilled identically:

$$\begin{aligned}
\sigma_r &= \frac{2\sigma_0}{\sqrt{3}}\left(\frac{1}{\rho}\phi' + \frac{1}{\rho^2}\phi^{\cdot\cdot}\right),\\
\sigma_\theta &= \frac{2\sigma_0}{\sqrt{3}}\phi'',\\
\tau_{r\theta} &= \frac{2\sigma_0}{\sqrt{3}}\left(\frac{1}{\rho}\phi^{\cdot\prime} - \frac{1}{\rho^2}\phi^{\cdot}\right),\\
\tau_{rz} &= \frac{2\sigma_0}{\sqrt{3}}\frac{1}{\rho}\psi^{\cdot},\\
\tau_{\theta z} &= -\frac{2\sigma_0}{\sqrt{3}}\psi',
\end{aligned} \tag{3.3}$$

where primes represent differentiation with respect to ρ and dots with respect to θ.

By using relations (2.3), as well as the incompressibility condition (2.4), a formula for the stress σ_z is obtained

$$\sigma_z = \frac{\sigma_0\sqrt{3}}{H}\varepsilon_z + \frac{1}{2}(\sigma_r + \sigma_\theta), \tag{3.4}$$

where the dimensionless modulus H proportional to φ was introduced for convenience

$$H = \frac{2\sigma_0}{\sqrt{3}}\varphi \tag{3.5}$$

The stress functions ϕ and ψ, as well as the modulus H, determine the state of stress and strain in the discussed problem.

Let us now substitute the relations (3.3) into the yield condition (2.2). After a few transformations and rearrangements, the first of the three required equations for ϕ, ψ and H is obtained as

$$\begin{aligned}
&\left\{\left(\phi'' - \frac{1}{\rho}\phi' - \frac{1}{\rho^2}\phi^{\cdot\cdot}\right)^2 + 4\left[\left(\frac{1}{\rho}\phi^{\cdot\prime} - \frac{1}{\rho^2}\phi^{\cdot}\right)^2 + (\psi')^2\right.\right.\\
&\qquad \left.\left. + \left(\frac{1}{\rho}\psi^{\cdot}\right)^2\right] - 1\right\}H^2 + 3(\kappa^2\rho^2\cos^2\theta + 2\kappa\rho\lambda\cos\theta + \lambda^2) = 0.
\end{aligned} \tag{3.6}$$

Another equation is obtained from the fourth compatibility condition (2.5) by performing the required differentiation and considering (2.3), (2.4) as well as (3.2) and (3.3)

$$\begin{aligned}
&\left(\rho^2\frac{\partial^2}{\partial\rho^2} + 3\rho\frac{\partial}{\partial\rho} - \frac{\partial^2}{\partial\theta^2}\right)\left[\left(\phi'' - \frac{1}{\rho}\phi' - \frac{1}{\rho^2}\phi^{\cdot\cdot}\right)H\right]\\
&\qquad + 4\left(\rho\frac{\partial^2}{\partial\rho\partial\theta} + \frac{\partial}{\partial\theta}\right)\left[\left(\frac{1}{\rho}\phi^{\cdot\prime} - \frac{1}{\rho^2}\phi^{\cdot}\right)H\right] = 0
\end{aligned} \tag{3.7}$$

The two last ones, not yet utilized conditions of compatibility (2.5), contain only strains γ_{rz} and $\gamma_{\theta z}$. Expressing them in terms of the functions ϕ and H, two equations are obtained:

$$\begin{aligned} \left(1-\rho\frac{\partial}{\partial\rho}-\rho^2\frac{\partial^2}{\partial\rho^2}\right)[H\psi']+\left(\frac{\partial}{\partial\theta}-\rho\frac{\partial^2}{\partial\rho\partial\theta}\right)\left[H\frac{1}{\rho}\psi^{\cdot}\right] &= 0, \\ \left(\rho\frac{\partial^2}{\partial\rho\partial\theta}\ \frac{\partial}{\partial\theta}\right)[H\psi']+\frac{\partial^2}{\partial\theta^2}\left[H\frac{1}{\rho}\psi^{\cdot}\right] &= 0 \end{aligned} \tag{3.8}$$

The principal determinant of the set of equations (3.8) equals zero since

$$\left(1-\rho\frac{\partial}{\partial\rho}-\rho^2\frac{\partial^2}{\partial\rho^2}\right)\frac{\partial^2}{\partial\rho^2}-\left(\rho\frac{\partial^2}{\partial\rho\partial\theta}+\frac{\partial}{\partial\theta}\right)\left(\frac{\partial}{\partial\theta}-\rho\frac{\partial^2}{\partial\rho\partial\theta}\right) = 0 \tag{3.9}$$

Thus the equations (3.8) may be written in the following way:

$$\begin{aligned} \left(1-\rho\frac{\partial}{\partial\rho}\right)\left\{\left(1+\rho\frac{\partial}{\partial\rho}\right)[H\psi']+\frac{\partial}{\partial\theta}\left[H\frac{1}{\rho}\psi^{\cdot}\right]\right\} &= 0, \\ \frac{\partial}{\partial\theta}\left\{\left(1+\rho\frac{\partial}{\partial\rho}\right)[H\psi']+\frac{\partial}{\partial\theta}\left[H\frac{1}{\rho}\psi^{\cdot}\right]\right\} &= 0, \end{aligned} \tag{3.10}$$

and the solution with respect to the expression in braces { } is:

$$\left(1+\rho\frac{\partial}{\partial\rho}\right)[H\psi']+\frac{\partial}{\partial\theta}\left[H\frac{1}{\rho}\psi^{\cdot}\right] = C\rho. \tag{3.11}$$

Constant C represents the dimensionless unit angle of twist, since

$$C\rho = \left(1+\rho\frac{\partial}{\partial\rho}\right)[-\varphi\tau_{\theta z}]+\frac{\partial}{\partial\theta}[\varphi\tau_{rz}], \tag{3.12}$$

and since for pure torsion

$$\gamma_{\theta z} = \rho\vartheta, \qquad \gamma_{rz} = 0,$$

then

$$C = -\vartheta \tag{3.13}$$

Finally, equation (3.11) becomes:

$$\left(1+\rho\frac{\partial}{\partial\rho}\right)[H\psi']+\frac{\partial}{\partial\theta}\left[H\frac{1}{\rho}\psi^{\cdot}\right]+\vartheta\rho = 0. \tag{3.14}$$

The system of non-linear, second or fourth order partial differential equations (3.6), (3.7), (3.14) determines the stress functions ϕ and ψ, as well as the plastic modulus H in the discussed problem.

4. SOLUTION FOR THE CASE OF HIGH INTERNAL PRESSURE

The solution of the basic system of equations (3.6), (3.7) and (3.14) is obtained by means of small parameter (perturbation) method, assuming a small effect of bending moment, axial force and torque on the limit carrying capacity of a pipe-line subjected to internal pressure. This method allows to reduce non-linear differential equations to simple linear equations; furthermore, apart from linearization it causes a separation of the system of equations.

We present namely the solution as series of following parameters: κ—proportional to curvature, λ—to unit extension, ϑ—to unit angle of twist. In the limit state of the tube considered, these parameters may be large, but finally we shall obtain series of parameters κ/D, λ/D and ϑ/D, where the constant D corresponds to internal pressure and is also suitably large. Thus κ/D, λ/D and ϑ/D may be considered as small.

We assume the solution in the form:

$$\begin{aligned} \phi &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \phi_{ijk}(\rho,\theta)\kappa^i\lambda^j\vartheta^k, \\ H &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} H_{ijk}(\rho,\theta)\kappa^i\lambda^j\vartheta^k, \\ \psi &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \psi_{ijk}(\rho,\theta)\kappa^i\lambda^j\vartheta^k, \end{aligned} \tag{4.1}$$

whilst in the case of pure internal pressure in plane strain condition we have:

$$\begin{aligned} \phi_{000} &= \tfrac{1}{2}\rho^2 \ln \rho - \tfrac{1}{4}\rho^2, \\ H_{000} &= \frac{D}{\rho^2}, \\ \psi_{000} &= 0, \end{aligned} \tag{4.2}$$

where D is an indeterminable constant in the fully plastic state. Equations (4.2) correspond to the classical circularly-symmetrical plastic solution. Much more complicated non-symmetrical plastic collapse modes were discussed in the paper by Życzkowski and the present author [7], but we do not use them here; they have no influence on the limit carrying capacity of the tube.

In order to solve the basic set of equations we begin with equation (3.14). Taking into account that the zero approximation of the function ψ equals zero, the subsequent functions ψ_{ijk} in terms of $H_{i-1,j,k}$, $H_{i,j-1,k}$ as well as $H_{i,j,k-1}$ can be obtained successively from this equation. Thus for particular corrections of function ψ we get equations of the type:

$$\psi''_{ijk} - \frac{1}{\rho}\psi'_{ijk} + \frac{1}{\rho^2}\ddot{\psi}_{ijk} = f_{ijk}(\rho,\theta). \tag{4.3}$$

The general solution of the above equation is assumed in the form of Fourier series:

$$\psi_{ijk} = \sum_{n=0}^{\infty} f_n(\rho)\cos n\theta + \bar{\psi}_{ijk}(\rho,\theta), \tag{4.4}$$

$\bar{\psi}_{ijk}(\rho,\theta)$ is a particular solution of the nonhomogeneous equation (4.3). Performing the required differentiations, the following equation for function $f_n(\rho)$ is obtained:

$$f''_n - \frac{1}{\rho}f'_n - \frac{n^2}{\rho^2}f_n = 0, \tag{4.5}$$

and then:

$$f_n(\rho) = C\rho^m, \qquad m = 1 \mp \sqrt{(1+n^2)}. \tag{4.6}$$

Substituting (4.6) into (4.3) we obtain finally:

$$\psi_{ijk} = \sum_{n=0}^{\infty} (C_{n1}\rho^{1-\sqrt{(1+n^2)}} + C_{n2}\rho^{1+\sqrt{(1+n^2)}}) \cos n\theta + \bar{\psi}_{ijk}(\rho, \theta). \tag{4.7}$$

Now, we shall proceed to the corrections of the function ϕ. Those can be obtained subsequently from equations (3.6). Note that by substituting ϕ_{000} and ψ_{000} into (3.6) the expression in brackets equals zero, thus

$$\phi''_{000} - \frac{1}{\rho}\phi'_{000} - \frac{1}{\rho^2}\ddot{\phi}_{000} = 1. \tag{4.8}$$

From this equation we can obtain successively ϕ_{ijk} in terms of function $H_{i-1,j,k}$, $H_{i,j-1,k}$, $H_{i,j,k-1}$ as well as $\psi_{i-1,j,k}$, $\psi_{i,j-1,k}$, $\psi_{i,j,k-1}$. In this way we obtain equations of the type

$$\phi''_{ijk} - \frac{1}{\rho}\phi'_{ijk} - \frac{1}{\rho^2}\ddot{\phi}_{ijk} = f_{ijk}(\rho, \theta). \tag{4.9}$$

The general solution is assumed, similarly as before, in the form of a series:

$$\phi_{ijk} = \sum_{n=0}^{\infty} f_n(\rho) \cos n\theta + \bar{\phi}_{ijk}(\rho, \theta); \tag{4.10}$$

and for the function $f_n(\rho)$ we have:

$$f''_n - \frac{1}{\rho} f'_n + \frac{n^2}{\rho^2} f_n = 0. \tag{4.11}$$

Upon solving the characteristic equation and substituting it into (4.10) the final solution is obtained:

$$\begin{aligned} \phi_{ijk} = {} & C_{01} + C_{02}\rho^2 + (C_{11}\rho + C_{12}\rho \ln \rho) \cos \theta \\ & + \sum_{n=2}^{\infty} [C_{n1}\rho \cos (\sqrt{(n^2-1)} \ln \rho) \\ & + C_{n2}\rho \sin (\sqrt{(n^2-1)} \ln \rho)] \cos n\theta + \bar{\phi}_{ijk}(\rho, \theta), \end{aligned} \tag{4.12}$$

where ϕ_{ijk} is a particular solution of nonhomogeneous equation (4.9).

In order to determine the functions H_{ijk}, let us consider equation (3.7). Substitution of ϕ_{000} into this equation will make the entire second term equal to zero and considering (4.8), H_{ijk} can be obtained in terms of $\phi_{i-1,j,k}$, $\phi_{i,j-1,k}$, $\phi_{i,j,k-1}$ by means of equations of the type:

$$H''_{ijk} + \frac{3}{\rho}H'_{ijk} - \frac{1}{\rho^2}\ddot{H}_{ijk} = f_{ijk}(\rho, \theta). \tag{4.13}$$

Assuming, as before, the general solution in form of a series

$$H_{ijk} = \sum_{n=0}^{\infty} f_n(\rho) \cos n\theta + \bar{H}_{ijk}(\rho, \theta); \tag{4.14}$$

and solving the equation with respect to function $f_n(\rho)$

$$f_n'' + \frac{3}{\rho} f_n' + \frac{n^2}{\rho^2} f_n = 0 \tag{4.15}$$

the following solutions for successive corrections of function H are finally obtained:

$$\begin{aligned} H_{ijk} &= C_{01} + \frac{C_{02}}{\rho^2} + \left(\frac{C_{11}}{\rho} + \frac{C_{12}}{\rho} \ln \rho\right) \cos \theta \\ &+ \sum_{n=2}^{\infty} \left[\frac{C_{n1}}{\rho} \cos (\sqrt{(n^2-1)} \ln \rho)\right. \\ &\left. + \frac{C_{n2}}{\rho} \sin (\sqrt{(n^2-1)} \ln \rho)\right] \cos n\theta + \bar{H}_{ijk}(\rho, \theta), \end{aligned} \tag{4.16}$$

where $\bar{H}_{ijk}(\rho, \theta)$ is a particular solution of nonhomogeneous equation (4.13).

Expressions (4.7), (4.12) and (4.16) determine the state of stress and strain in the problem under consideration. Now we shall apply the boundary conditions.

On the outer radius we expect zero stresses:

$$\rho = 1; \qquad \sigma_r = \tau_{r\theta} = \tau_{rz} = 0; \tag{4.17}$$

introducing stress functions (3.3) we have:

$$\begin{aligned} \phi'_{ijk}(1) + \phi^{\cdot\cdot}_{ijk}(1) &= 0, \\ \phi^{\cdot\prime}_{ijk}(1) - \phi^{\cdot}_{ijk}(1) &= 0, \\ \psi^{\cdot}_{ijk}(1) &= 0 \end{aligned} \tag{4.18}$$

On the inner radius only conditions in integral form could be fulfilled:

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \sigma_r \, d\theta &= -p, \\ \frac{1}{2\pi} \int_0^{2\pi} \tau_{r\theta} \, d\theta &= 0, \end{aligned} \tag{4.19}$$

where p is an internal pressure existing in the tube.

Most of the conditions (4.19) are automatically fulfilled in most cases, and the number of equations (4.18) is as a rule not sufficient to determine all constants in (4.7), (4.12) and (4.16). In particular cases without bending, the condition of circular symmetry can be applied

$$\phi^{\cdot}_{ijk} = H^{\cdot}_{ijk} = \psi^{\cdot}_{ijk} = 0, \tag{4.20}$$

and the remaining coefficients can be obtained from displacement conditions

$$r\varepsilon_\theta - u_r = 0, \tag{4.21}$$

because

$$u_\theta = \int (\varepsilon_\theta r - u_r) \, d\theta + C_\theta = u_\theta(r) \tag{4.22}$$

Using the formulae (4.7), (4.22) and (4.16), and considering boundary conditions (4.18)–(4.21), corrections of the functions ϕ, ψ and H can be obtained successively. Some of them equal zero, as it can easily be shown, so that, confining ourselves to the terms of the second order only, we have:

$$
\begin{aligned}
\phi_{100} &= \phi_{010} = \phi_{001} = \phi_{011} = \phi_{101} = 0,\\
H_{100} &= H_{010} = H_{001} = H_{011} = H_{101} = 0,\\
\psi_{000} &= \psi_{100} = \psi_{010} = \psi_{110} = \psi_{101} = 0.
\end{aligned}
\tag{4.23}
$$

For the other corrections different from zero we obtain:

$$
\begin{aligned}
\phi_{200} &= C_{01}+\frac{1}{16D^2}\rho^2-\frac{1}{64D^2}\rho^8+C_{11}\rho\cos\theta\\
&\quad+\frac{3}{208D^2}\left[\rho\cos(\sqrt{3}\ln\rho)+\frac{7}{\sqrt{3}}\rho\sin(\sqrt{3}\ln\rho)-\rho^8\right]\cos 2\theta,\\
\phi_{020} &= C_{01}+\frac{3}{16D^2}\rho^2-\frac{1}{16D^2}\rho^6,\\
\phi_{002} &= C_{01}-\frac{1}{24D^2}\rho^2-\frac{1}{96D^2}\rho^8,\\
\phi_{110} &= C_{01}+\left(C_{11}\rho+\frac{1}{2D^2}\rho\ln\rho-\frac{1}{12D^2}\rho^7\right)\cos\theta,\\
\psi_{001} &= C_{01}-\frac{1}{8D}\rho^4,
\end{aligned}
\tag{4.24}
$$

as well as:

$$
\begin{aligned}
H_{002} &= \frac{C_{02}}{\rho^2}+\frac{\rho^4}{2D},\\
H_{020} &= C_{01}+\frac{C_{02}}{\rho^2}+\frac{3}{2D}\rho^2,\\
H_{200} &= C_{01}+\frac{C_{02}}{\rho^2}+\frac{3}{4D}\rho^4+\frac{C_{11}}{\rho}\cos\theta\\
&\quad+\frac{3}{13D}\left[2\rho^4-\frac{2}{\rho^3}\cos(\sqrt{3}\ln\rho)+\frac{3\sqrt{3}}{4\rho^3}\sin(\sqrt{3}\ln\rho)\right]\cos 2\theta\\
&\quad+\sum_{n=3}^{\infty}\left[\frac{C_{n1}}{\rho}\cos(\sqrt{(n^2-1)}\ln\rho)+\frac{C_{n2}}{\rho}\sin(\sqrt{(n^2-1)}\ln\rho)\right]\cos n\theta,\\
H_{110} &= C_{01}+\frac{C_{02}}{\rho^2}+\left(\frac{5}{2D}\rho^3+\frac{1}{D\rho^3}+\frac{C_{11}}{\rho}\right)\cos\theta\\
&\quad+\sum_{n=2}^{\infty}\left[\frac{C_{n1}}{\rho}\cos(\sqrt{(n^2-1)}\ln\rho)+\frac{C_{n2}}{\rho}\sin(\sqrt{(n^2-1)}\ln\rho)\right]\cos n\theta
\end{aligned}
\tag{4.25}
$$

Constants C_{01}, $C_{02}, \ldots C_{n1}$, C_{n2} in (4.25) remain indetermined because there are no boundary conditions for function H. Some of them could be determined by calculating the corrections of higher order for functions ϕ and ψ and applying suitable conditions (4.18).

Based on expressions (4.24), and considering (3.3) and (3.4), we are able to determine the stresses. Constants C_{01} and C_{11} have no effect on the distribution of stresses, which can be presented as follows

$$
\begin{aligned}
\sigma_r &= \frac{2\sigma_0}{\sqrt{3}}\Bigg[\ln\rho + \Bigg\{\tfrac{1}{8}(1-\rho^6) + \tfrac{3}{52}\Bigg[\frac{1}{\rho}\cos(\sqrt{3}\ln\rho) - \frac{2\sqrt{3}}{\rho}\sin(\sqrt{3}\ln\rho) \\
&\quad -\rho^6\Bigg]\cos 2\theta\Bigg\}\frac{\kappa^2}{D^2} + \tfrac{3}{8}(1-\rho^4)\frac{\lambda^2}{D^2} + \tfrac{1}{12}(1-\rho^6)\frac{\vartheta^2}{D^2} \\
&\quad + \frac{1}{2}\left(\frac{1}{\rho} - \rho^5\right)\cos\theta\frac{\kappa\lambda}{D^2} + \ldots\Bigg], \\
\sigma_\theta &= \frac{2\sigma_0}{\sqrt{3}}\Bigg[\ln\rho + 1 + \Bigg\{\tfrac{1}{8}(1-7\rho^6) + \tfrac{3}{52}\Bigg[\frac{1}{\rho}\cos(\sqrt{3}\ln\rho) \\
&\quad - \frac{2\sqrt{3}}{\rho}\sin(\sqrt{3}\ln\rho) - 14\rho^6\Bigg]\cos 2\theta\Bigg\}\frac{\kappa^2}{D^2} + \tfrac{3}{8}(1-5\rho^4)\frac{\lambda^2}{D^2} \\
&\quad + \tfrac{1}{12}(1-7\rho^6)\frac{\vartheta^2}{D^2} + \tfrac{1}{2}\left(\frac{1}{\rho} - 7\rho^5\right)\cos\theta\frac{\kappa\lambda}{D^2} + \ldots\Bigg], \qquad (4.26) \\
\sigma_z &= \frac{2\sigma_0}{\sqrt{3}}\Bigg[\ln\rho + \tfrac{1}{2} + \tfrac{3}{2}\rho^3\cos\theta\frac{\kappa}{D} + \tfrac{3}{2}\rho^2\frac{\lambda}{D} \\
&\quad + \Bigg\{\tfrac{1}{8}(1-4\rho^6) + \tfrac{3}{52}\Bigg[\frac{1}{\rho}\cos(\sqrt{3}\ln\rho) - \frac{2\sqrt{3}}{\rho}\sin(\sqrt{3}\ln\rho) \\
&\quad - \tfrac{15}{2}\rho^6\Bigg]\cos 2\theta\Bigg\}\frac{\kappa^2}{D^2} + \tfrac{3}{8}(1-3\rho^4)\frac{\lambda^2}{D^2} + \tfrac{1}{12}(1-4\rho^6)\frac{\vartheta^2}{D^2} \\
&\quad + \tfrac{1}{2}\left(\frac{1}{\rho} - 4\rho^5\right)\sin\theta\frac{\kappa\lambda}{D^2} + \ldots\Bigg], \\
\tau_{r\theta} &= \frac{2\sigma_0}{\sqrt{3}}\Bigg[\frac{3}{104}\Bigg[\frac{7}{\rho}\cos(\sqrt{3}\ln\rho) - \frac{\sqrt{3}}{\rho}\sin(\sqrt{3}\ln\rho) - 7\rho^6\Bigg] \\
&\quad \times \sin 2\theta\frac{\kappa^2}{D^2} + \frac{1}{2}\left(\frac{1}{\rho} - \rho^5\right)\sin\theta\frac{\kappa\lambda}{D^2} + \ldots\Bigg], \\
\tau_{\theta z} &= \frac{2\sigma_0}{\sqrt{3}}\left[\frac{1}{2}\rho^3\frac{\vartheta}{D} + \ldots\right], \\
\tau_{zr} &= 0 + \ldots.
\end{aligned}
$$

Parameters κ, λ, ϑ in expressions (4.26) do not appear independently but in the form of relations κ/D, λ/D and ϑ/D where D is the indeterminable constant from (4.2). From now on we shall use new relations for the sake of convenience:

$$\kappa/D = \hat{\kappa}, \qquad \lambda/D = \hat{\lambda}, \qquad \vartheta/D = \hat{\vartheta}. \tag{4.27}$$

Applying (4.25)–(4.26) and introducing physical relations (2.3), formulae for strains could also be obtained, remembering that in our case the strain ε_z is given by (3.2).

5. LOAD ANALYSIS: INTERACTION (LIMIT) SURFACE

Now we proceed to determine the external loads which cause the fully plastic state of the tube.

The axial force can be obtained from formulae:

$$N = \iint_F \sigma_z \, \mathrm{d}F = b^2 \int_0^{2\pi} \mathrm{d}\theta \int_\beta^1 \sigma_z \rho \, \mathrm{d}\rho, \tag{5.1}$$

(F being the cross-sectional area of the tube), and introducing dimensionless values and taking (4.26) into consideration:

$$n = \frac{\sqrt{3}}{2\pi\sigma_0 b^2} N = \beta^2 \ln\frac{1}{\beta} + \tfrac{3}{4}(1-\beta^4)\hat{\lambda} - \tfrac{1}{8}\beta^2(1-\beta^6)\hat{\kappa}^2 \tag{5.2}$$
$$- \tfrac{3}{8}\beta^2(1-\beta^4)\hat{\lambda}^2 - \tfrac{1}{12}\beta^2(1-\beta^6)\hat{\vartheta}^2 + \ldots.$$

The dimensionless axial force $\beta^2 \ln 1/\beta$ relates to the pure internal pressure in case of plane strain. It will be more convenient to use a reduced value of the axial force:

$$\bar{n} = n - \beta^2 \ln\frac{1}{\beta} = \tfrac{3}{4}(1-\beta^4)\hat{\lambda} - \tfrac{1}{8}\beta^2(1-\beta^6)\hat{\kappa}^2 \tag{5.3}$$
$$- \tfrac{3}{8}\beta^2(1-\beta^4)\hat{\lambda}^2 - \tfrac{1}{12}\beta^2(1-\beta^6)\hat{\vartheta}^2 + \ldots,$$

which, assuming small parameters $\hat{\kappa}$, $\hat{\lambda}$, $\hat{\vartheta}$, will also be small. The bending moment is calculated from formulae:

$$M_g = \iint_F r\sigma_z \cos\theta \, \mathrm{d}F = b^3 \int_\beta^1 \rho^2 \, \mathrm{d}\rho \int_0^{2\pi} \sigma_z \cos\theta \, \mathrm{d}\theta, \tag{5.4}$$

from which, considering (4.26) and performing required integrations, we obtain a dimensionless expression:

$$m_g = \frac{2\sqrt{3}}{\pi\sigma_0 b^3} M_g = (1-\beta^6)\hat{\kappa} - \beta^2(1-\beta^6)\bar{n}\hat{\lambda} + \ldots \tag{5.5}$$

Finally the torque

$$M_s = \iint_F \tau_{\theta z} r \, \mathrm{d}F = 2\pi b^3 \int_\beta^1 \tau_{\theta z} \rho^2 \, \mathrm{d}\rho, \tag{5.6}$$

or, in dimensionless form:

$$m_s = \frac{\sqrt{3}}{W_0 \sigma_0} M_s = \frac{2}{3}\frac{1-\beta^6}{1-\beta^4}\hat{\vartheta} + \ldots, \tag{5.7}$$

where W_0 is the elastic section modulus in torsion

$$W_0 = \frac{\pi b^3(1-\beta^4)}{2} \tag{5.8}$$

Some complications are connected with calculation of internal pressure. The assumption of full plastification of the whole cross section of the tube leads to variable normal stresses σ_r at the internal radius β. Also, tangential stresses $\tau_{r\theta}$ are not strictly zero along the inner perimeter. So we arrive at the problem of how to apply the results obtained by means of the perturbation method (4.26) for variable pressure along the inner perimeter to a pipeline loaded by constant normal internal pressure. It can be done in various approximate ways.

The simplest way is to take mean values of the obtained expressions for stresses σ_r and $\tau_{r\theta}$. The internal pressure is now calculated as follows:

$$p_m = -\frac{1}{2\pi}\int_0^{2\pi} \sigma_r|_{\rho=\beta}\, \mathrm{d}\theta \tag{5.9}$$

where p_m indicates mean value of the pressure. Introducing the dimensionless pressure q_m and performing the required integration we have

$$q_m = \frac{\sqrt{3}}{2\sigma_0}p_m = \ln\frac{1}{\beta} - \tfrac{1}{8}(1-\beta^6)\hat{\kappa}^2 - \tfrac{3}{8}(1-\beta^4)\hat{\lambda}^2 - \tfrac{1}{12}(1-\beta^6)\hat{\vartheta}^2 + \dots \tag{5.10}$$

Expressions (5.2), (5.5), (5.7) and (5.10) represent the limit surface in parametrical way, the parameters being $\hat{\kappa}$, $\hat{\lambda}$, $\hat{\vartheta}$ and, additionally, the ratio of the radii β.

Taking instead of the mean value of p the lower bound of $(-\sigma_r)$ at $\rho = \beta$ we may obtain a lower bound of the internal pressure. Such an approach was used in the paper by Życzkowski [11].

The obtained general integrals may also be applied to the formulated problem using the Trefftz method, since the differential equations are here satisfied exactly in the sense of the perturbation method and there exists some difficulty with the boundary conditions. Instead of previously used boundary conditions (4.17) and (4.19) we shall now introduce the condition of least square deviations, namely, minimum of the integral expression

$$\begin{aligned} F(C_{ij}) = &\int_0^{2\pi} \{\sigma_{rT}(1)-\sigma_r(1)]^2+[\tau_{r\theta T}(1)-\tau_{r\theta}(1)]^2\}\, \mathrm{d}\theta \\ &+\int_0^{2\pi} \{[\sigma_{rT}(\beta)-\sigma_r(\beta)]^2+[\tau_{r\theta T}(\beta)-\tau_{r\theta}(\beta)]^2\}\, \mathrm{d}\theta = \min., \end{aligned} \tag{5.11}$$

where indices T indicate the solutions obtained by the Trefftz method and $\sigma_r(1)$, $\tau_{r\theta}(1)$, $\sigma_r(\beta)$, $\tau_{r\theta}(\beta)$ are determined by actual exact boundary conditions, namely,

$$\begin{aligned} \sigma_r(1) &= \tau_{r\theta}(1) = \tau_{r\theta}(\beta) = 0 \\ \sigma_r(\beta) &= -p = \text{const.} \end{aligned} \tag{5.12}$$

Introducing (4.12) and (5.12) into the formula (5.11) and applying the condition of the minimum of the function $F(C_{ij})$, we obtain

$$\sigma_{rT} = \frac{2Q}{\sqrt{3}}\left[\ln\rho + \left\{\frac{1-\rho^2}{8} + \left[\frac{-3\bar{C}_{21}+\sqrt{3}\bar{C}_{22}}{\rho}\cos(\sqrt{3}\ln\rho) - \frac{3\bar{C}_{22}+\sqrt{3}\bar{C}_{21}}{\rho}\sin(\sqrt{3}\ln\rho)\right.\right.\right.$$
$$\left.\left. - \tfrac{3}{52}\rho^6\right]\cos 2\theta\right\}\hat{\kappa}^2 + \frac{3(1-\rho^4)}{8}\hat{\lambda}^2 + \frac{1-\rho^6}{12}\hat{\vartheta}^2 + \tag{5.13}$$
$$\left. + \left\{\frac{1}{2}\left[\frac{\beta(1-\beta^5)}{1+\beta}\frac{1}{\rho} - \rho^5\right]\cos\theta\right\}\hat{\kappa}\hat{\lambda} + \dots\right].$$

Constants $\bar{C}_{21}$ and $\bar{C}_{22}$ may be calculated from the conditions

$$\frac{\partial F}{\partial \bar{C}_{21}} = 0, \qquad \frac{\partial F}{\partial \bar{C}_{22}} = 0, \tag{5.14}$$

They have, however, no influence on the loadings. Expressions for the other stresses σ_θ, σ_z and $\tau_{r\theta}$ may be obtained in a similar way.

Introducing, as previously, dimensionless notations and integrating, we can now determine the loads. Internal pressure q_T, axial force n_T and torque m_{sT} are exactly the same as in formulae (5.3), (5.7) and (5.10),

$$\begin{aligned} q_T &= q_m \\ n_T &= n \\ m_{sT} &= m_s \end{aligned} \tag{5.15}$$

Only for the bending moment do we obtain a slightly different formula

$$m_{gT} = (1-\beta^6)\hat{\kappa} - [(1-\beta^8) - \beta(1-\beta)(1+\beta^5)]\hat{\kappa}\hat{\lambda} + \dots \tag{5.16}$$

Expressions (5.15) and (5.16) represent the parametrical equation of the limit surface obtained using the Trefftz method, the parameters being as previously κ, $\hat{\lambda}$, $\hat{\vartheta}$ and ratio of radii β.

For some practical purposes it may be more convenient to determine the limit surface by the equation in an explicit form

$$q = q(\bar{n}, m_s, m_g), \tag{5.17}$$

which is obtained by eliminating the parameters $\hat{\kappa}$, $\hat{\lambda}$, $\hat{\vartheta}$ from the obtained system of equations.

For this reason let us invert the series and write parameters $\hat{\kappa}$, $\hat{\lambda}$, $\hat{\vartheta}$ in the form

$$\begin{aligned} \hat{\kappa} &= \kappa_{100}m_g + \kappa_{010}\bar{n} + \kappa_{001}m_s + \kappa_{110}m_g\bar{n} + \kappa_{011}\bar{n}m_s \\ &\quad + \kappa_{101}m_g m_s + \kappa_{200}m_g^2 + \kappa_{020}\bar{n}^2 + \kappa_{002}m_s^2 + \dots \\ \hat{\lambda} &= \lambda_{100}m_g + \lambda_{010}\bar{n} + \lambda_{001}m_s + \lambda_{110}m_g\bar{n} + \lambda_{101}m_g m_s \\ &\quad + \lambda_{200}m_g^2 + \lambda_{020}\bar{n}^2 + \lambda_{002}m_s^2 + \dots \\ \hat{\vartheta} &= \vartheta_{100}m_g + \vartheta_{010}\bar{n} + \vartheta_{001}m_s + \vartheta_{110}m_g\bar{n} + \vartheta_{011}\bar{n}m_s \\ &\quad + \vartheta_{101}m_g m_s + \vartheta_{200}m_g^2 + \vartheta_{020}\bar{n}^2 + \vartheta_{002}m_s^2 + \dots \end{aligned} \tag{5.18}$$

Substituting expressions (5.18) into the obtained parametrical equations we can obtain all coefficients κ_{ijk}, λ_{ijk}, ϑ_{ijk}. For example, using the equations (5.3), (5.5), (5.7) and (5.10), we obtain:

$$
\begin{aligned}
&\lambda_{100} = \lambda_{001} = \lambda_{011} = \lambda_{110} = \lambda_{101} = 0,\\
&\lambda_{010} = \frac{4}{3(1-\beta^4)}, \qquad \lambda_{200} = \frac{1}{6}\frac{\beta^2}{(1-\beta^4)(1-\beta^6)},\\
&\lambda_{020} = \frac{8}{9}\frac{\beta^2}{(1-\beta^4)^2}, \qquad \lambda_{002} = \frac{1}{4}\frac{\beta^2(1-\beta^4)}{(1-\beta^6)},\\
&\kappa_{010} = \kappa_{001} = \kappa_{011} = \kappa_{110} = \kappa_{200} = \kappa_{020} = \kappa_{002} = 0,\\
&\kappa_{100} = \frac{1}{1-\beta^6}, \qquad \kappa_{110} = \frac{4}{3}\frac{\beta^2}{(1-\beta^4)(1-\beta^6)},\\
&\vartheta_{100} = \vartheta_{010} = \vartheta_{110} = \vartheta_{011} = \vartheta_{101} = \vartheta_{200} = \vartheta_{020} = \vartheta_{002} = 0,\\
&\vartheta_{001} = \frac{3}{2}\frac{1-\beta^4}{1-\beta^6}.
\end{aligned}
\tag{5.19}
$$

Substituting (5.19) into (5.18) we have:

$$
\begin{aligned}
\hat{\lambda} &= \frac{4}{3(1-\beta^4)}\bar{n} + \frac{1}{6}\frac{\beta^2}{(1-\beta^4)(1-\beta^6)}m_g^2 + \frac{8}{9}\frac{\beta^2}{(1-\beta^4)^2}\bar{n}^2\\
&\quad + \frac{1}{4}\beta^2\frac{1-\beta^4}{1-\beta^6}m_s^2 + \ldots,\\
\hat{\kappa} &= \frac{1}{1-\beta^6}m_g + \frac{4}{3}\frac{\beta^2}{(1-\beta^4)(1-\beta^6)}m_g\bar{n} + \ldots,\\
\hat{\vartheta} &= \frac{3}{2}\frac{1-\beta^4}{1-\beta^6}m_s + \ldots,
\end{aligned}
\tag{5.20}
$$

and substituting further (5.10), with (5.3) taken into consideration, we obtain the required final form of the limit surface equation:

$$
\begin{aligned}
q_m &= \ln\frac{1}{\beta} - \frac{1}{8(1-\beta^6)}m_g^2 - \frac{2}{3(1-\beta^4)}\left(n-\beta^2\ln\frac{1}{\beta}\right)^2\\
&\quad - \frac{3(1-\beta^4)^2}{16(1-\beta^6)}m_s^2 - \frac{8\beta^2}{9(1-\beta^4)^2}\left(n-\beta^2\ln\frac{1}{\beta}\right)^3\\
&\quad - \frac{\beta^2}{2(1-\beta^4)(1-\beta^6)}\left(n-\beta^2\ln\frac{1}{\beta}\right)m_g^2\\
&\quad - \frac{\beta^2(1-\beta^4)}{4(1-\beta^6)}\left(n-\beta^2\ln\frac{1}{\beta}\right)m_s^2 - \ldots
\end{aligned}
\tag{5.21}
$$

Using the Trefftz method (5.15), (5.16) we obtain finally a similar equation

$$q_T = \ln\frac{1}{\beta} - \frac{1}{8(1-\beta^4)}m_g^2 - \frac{2}{3(1-\beta^4)}\left(n-\beta^2\ln\frac{1}{\beta}\right)^2 - \frac{3(1-\beta^4)^2}{16(1-\beta^6)}m_s^2$$
$$-\frac{8\beta^2}{9(1-\beta^4)^2}\left(n-\beta^2\ln\frac{1}{\beta}\right)^3 - \frac{1}{3(1-\beta^4)(1-\beta^6)}\left[\frac{\beta^2}{2} - \frac{1-\beta^8-\beta(1-\beta)(1+\beta^5)}{1-\beta^6}\right] \quad (5.22)$$
$$\times\left(n-\beta^2\ln\frac{1}{\beta}\right)m_g^2 - \frac{\beta^2(1-\beta^4)}{4(1-\beta^6)}\left(n-\beta^2\ln\frac{1}{\beta}\right)m_s^2 - \dots$$

Equations (5.21) and (5.22) determine the limit surface for a tube in an explicit way: (5.21) for mean approximation, (5.22) for the Trefftz method of fulfillment of boundary conditions. They may be applied immediately to determine the limit carrying capacity of an isostatic pipe-line; in the case of a hyperstatic one they may be used to find possible plastic collapse modes.

The sections of the limit surface are shown in Fig. 2.

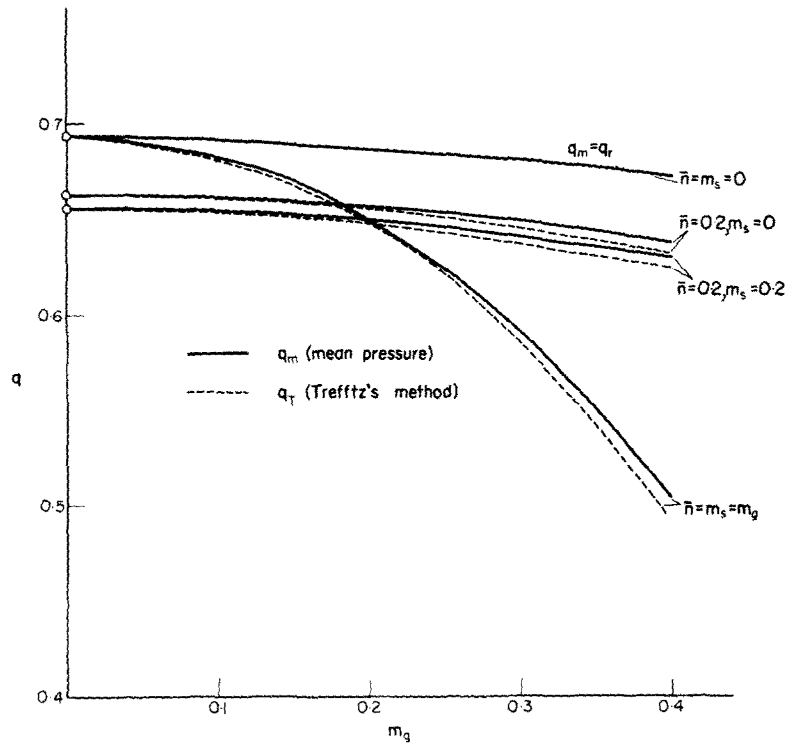

FIG. 2

6. AN EXAMPLE AND A DISCUSSION OF CORRECTNESS OF THE SOLUTION

The accepted assumption of reaching the limit carrying capacity at the full plastification of the whole section leads in consequence to incorrectness in fulfilling the boundary conditions for stresses σ_r and $\tau_{r\theta}$. Using, for example, the first way described in Section 5, at the inner radius only the integral conditions (4.19) are fulfilled.

Let us now analyze the order of discrepancy between the mean value of stress σ_r and its real value at the inner radius, and calculate the deviation of stress $\tau_{r\theta}$ from zero. Calculation will be done for ratio of radii $\beta = 0{\cdot}5$, parameters $\hat{\kappa} = \hat{\lambda} = \hat{\vartheta} = 0{\cdot}1$, and corresponding values of loads:

$$\bar{n} = 0{\cdot}0689, \quad m_g = 0{\cdot}0960, \quad m_s = 0{\cdot}0934. \tag{6.1}$$

The real distribution of stresses σ_r and $\tau_{r\theta}$ at the inner radius is obtained from formulae (4.26), whereas the mean value of pressure is obtained from (5.10). The results are shown in Fig. 3.

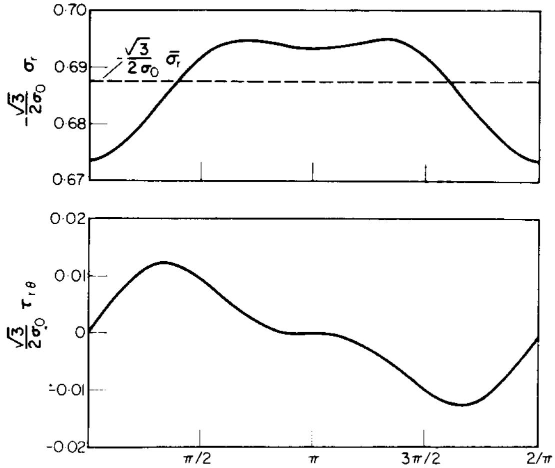

FIG. 3

Taking the difference between the real and the mean value of pressure as a measure of error, it is found that in our case the error does not exceed 2 percent. When the parameters $\hat{\kappa}$ and $\hat{\lambda}$ increase the value of the error also increases, but—on the other side—for pure bending, pure torsion etc., is equal zero again.

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Абстракт—Целью работы является определение предельного состояния трубы, подверженной четырёхкратной комбинации несимметрических нагрузок: внутреннему давлению, кручению, изгибающему моменту и осевой силе. Система всех шестнадцати уравений теории пластичности сводится к трем, путем введения дьух функций напряжения ϕ и ψ и модуля пластичности H. Система решается методом тройных возмущений, при предположении, что изгиб, кручение и растяжение, вызывают небольшое влияние на исчерпание несущей способности трубы, подверженной внутреннему давлению.